Path Integral Approach to Quantum Mechanics

Emilia Szymańska, University of Wrocław, Faculty of Physics and Astronomy
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We present the path integral approach to quantum mechanics and show its equivalence to the Schrödinger picture. We apply the method to a general case, a free particle and harmonic oscillator. Then we discuss propagators in imaginary time and many possibilities and insight brought by this change.

I. INTRODUCTION

The fundamental problem we try to solve in quantum mechanics is to find the time evolution of a state. In order to do that multiple approaches have been developed in the early twentieth century. In 1926 Erwin Schrödinger published a paper [1] in which he postulated a differential equation which was to govern the wave function of a quantum mechanical system. In Schrödinger's formulation the Hamiltonian and its eigenstates are in the limelight. Their time evolution is easy to find as they are stationary. Another well known approach was introduced a year earlier by Werner Heisenberg who based his formulation on matrix algebra, shifting the time dependence to the operators.

We know that in classical mechanics the Lagrangian and Hamiltonian approaches are proven to be equivalent. Hence, a question arises: can we construct a formulation based on Lagrangians which would be equivalent to Schrödinger's? In 1932 Paul Dirac published a paper in which he had shown how Lagrangians appear naturally in quantum mechanics. He defined a transition probability amplitude as an inner product of a Schrödinger picture wave function evaluated at a starting point and a possible future point. This has inspired Feynman to seek a correlation between this probability amplitude and the exponent of classical action.

It wasn't until 1948 than Feynman finally formalised his intuition in his paper [2]. His idea was that the time evolution of a state can be found by summing Dirac's transition probability amplitudes over all possible points of the trajectory of a particle:

$$|\Psi(x,t)\rangle = \int_{-\infty}^{\infty} \langle \Psi(x',t) | \Psi(x_i,t_i)\rangle \, dx' \, |\Psi(x',t)\rangle \tag{1}$$

In Zee's well known QFT textbook it is presented as a solution of double slit experiment expanded into infinitely many screens with infinitely many slits [3].

We're going to derive a propagator for a general case. Then we will fix the normalization with the help of the propagator for a free particle. We will also apply the path integral method to a simple harmonic oscillator. At the end we will discuss the advantages and drawbacks of the path integral approach.

II. GENERAL CASE

A propagator specifies the probability amplitude for a particle to travel from one point to another in a given time or to be travelling with specific momentum and energy. It is calculated by taking an inner product of Heisenberg picture wave functions evaluated at the starting point and one evaluated at a possible future point. One can understand it as the probability amplitude of a state to evolve in a specific way in a specific time interval.

First, we would like to calculate the propagator for a general case: $H = \frac{\hat{p}^2}{2m} + V(\hat{x})$. The quantity we want to obtain is

$$\begin{aligned}\mathcal{K} &= \langle x_f, t_f | x_i, t_i \rangle \\ &= \langle x_f | \, e^{-\frac{i}{\hbar}H(t_f - t_i)} \, | x_i \rangle \\ &= \int dx_{N-1} \dots dx_1 \int dp_{N-1} \dots dp_0 \\ &\quad \times \langle x_f | p_{N-1}\rangle \langle p_{N-1} | \, e^{-\frac{i}{\hbar}H\frac{t_f - t_i}{\hbar}} \, | x_{N-1}\rangle \dots \end{aligned} \tag{2}$$

where we have inserted resolution of identity $N-1$ times. Let's find an element of the integrand sandwiched between $(N-1)$th states

$$\begin{aligned}\mathcal{K}_{N-1} &= \langle p_{N-1} | \, e^{-\frac{i}{\hbar}H\Delta t} \, | x_{N-1}\rangle \\ &= \langle p_{N-1} | \, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\Delta t} e^{-\frac{i}{\hbar}V(\hat{x})\Delta t} \, | x_{N-1}\rangle\end{aligned}$$

where $\Delta t = \frac{t_f - t_i}{N}$. We used Baker-Campbell-Hausdorff formula and the fact that

$$e^{-\frac{1}{2}[\hat{p}^2, V(\hat{x})]} = 1 + \mathcal{O}[(\Delta t)^2] \approx 1$$

for small Δt. We replace the operators with their eigenvalues

$$\begin{aligned}\mathcal{K}_{N-1} &= \langle p_{N-1} | \, e^{-\frac{i}{\hbar}\frac{p_{N-1}^2}{2m}\Delta t} e^{-\frac{i}{\hbar}V(x_{N-1})\Delta t} \, | x_{N-1}\rangle \\ &= \exp\left[-\frac{i}{\hbar}\left(\frac{p_{N-1}^2}{2m} + V(x_{N-1})\right)\Delta t\right] \langle p_{N-1} | x_{N-1}\rangle\end{aligned}$$

We use this to calculate an important part of $\mathcal{K}$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dp_{N-1} \langle x_f | p_{N-1} \rangle \mathcal{K}_{N-1} = \\
&= \int_{-\infty}^{\infty} dp_{N-1} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_{N-1} x_f} \\
&\quad \times \exp\left(-\frac{i}{\hbar}\left(\frac{p_{N-1}^2}{2m} + V(x_{N-1})\right)\Delta t\right) \langle x_{N-1} | p_{N-1} \rangle \\
&= \frac{1}{2\pi\hbar}\left[\int_{-\infty}^{\infty} dp_{N-1} e^{-\frac{i\Delta t}{2\hbar m} p_{N-1}^2 + \frac{i}{\hbar}(x_f - x_{N-1}) p_{N-1}}\right] \\
&\qquad\qquad \times e^{-\frac{i}{\hbar} V(x_{N-1})\Delta t}
\end{aligned}$$

remembering that $\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} px}$. We recognize this expression as an easily soluble Gaussian integral. We get then

$$\begin{aligned}
\mathcal{K}_{N-1} =& \sqrt{\frac{-im}{2\pi\hbar} \frac{1}{\Delta t}} \\
& \times \exp\left[-\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{x_f - x_{N-1}}{\Delta t}\right)^2 + V(x_{N-1})\right)\right]
\end{aligned} \tag{3}$$

We can finally plug (3) into (2). After doing some integrations we see a pattern which leads us to

$$\begin{aligned}
\mathcal{K} =& \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 \dots dx_{N-1} \left(\frac{-im}{2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \\
& \times \prod_{i=1}^{N-1} \exp\left(-\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 + V(x_i)\right)\Delta t\right) \\
=& \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 \dots dx_{N-1} \left(\frac{-im}{2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \\
& \times \exp\left[\sum_{i=1}^{N-1}\left(-\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 + V(x_i)\right)\right)\Delta t\right]
\end{aligned}$$

The explicit derivation of the above can be found in Feynman and Hibbs [4]. Next we would like to take the limit $N \to \infty$ but one thing is bothering us. We know that the factor of $\sqrt{\frac{1}{(2\pi i)^N}}$ will not look pretty in the limit. The only way we can secure the convergence is to fix the normalization in a proper way. We will do that later in the text (9). As a result our propagator is

$$\mathcal{K} = \int_{-\infty}^{\infty} D[x(t)]\, e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right)dt} \tag{4}$$

where we defined

$$\begin{aligned}
\int_{-\infty}^{\infty} D[x(t)] = \lim_{N\to\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 \dots dx_{N-1} \\
\times \left(\frac{-im}{2\pi\hbar\Delta t}\right)^{\frac{N}{2}}
\end{aligned}$$

III. EQUIVALENCE TO SCHRÖDINGER'S EQUATION

Now that we are familiar with the new method we must ask ourselves if it really is a picture equivalent to non-relativistic quantum mechanics we know. Let's attempt to recreate Schrödinger's equation starting out from the path integral

$$\langle x_f, t_f | x_i, t_i \rangle = \int D[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]} \tag{5}$$

Let's now shift the trajectory by a small amount δx, keeping x_i fixed (so that $\delta x_i = 0$) and varying x_f. Let's see how both sides of (5) change. Left hand side:

$$\begin{aligned}
&\langle x_f + \delta x_f, t_f | x_i, t_i \rangle - \langle x_f, t_f | x_i, t_i \rangle \\
&= \frac{\partial}{\partial x_f} \langle x_f, t_f | x_i, t_i \rangle\, \delta x(t_f)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\int D[x(t)]\, e^{\frac{i}{\hbar} S[x(t)+\delta x(t)]} - \int D[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]} \\
&= \int Dx(t)\, e^{\frac{i}{\hbar} S[x(t)]} \frac{i\delta S}{\hbar}
\end{aligned} \tag{6}$$

From Classical Mechanics we know that the action changes by

$$\begin{aligned}
\delta S &= S[x(t) - \delta x(t)] - S[x(t)] \\
&= \int_{t_i}^{t_f} dt \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x}\right) \\
&= \frac{\partial \mathcal{L}}{\partial \dot{x}}\, \delta x(t)\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f}\left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta x
\end{aligned}$$

Recalling Euler-Lagrange equation we see that the second part is equal to zero. We are thus left with

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x(t_f) = p(t_f)\delta x(t_f)$$

knowing that $p(t) = \frac{\partial \mathcal{L}}{\partial \dot{x}}$. Dropping the $\delta x(t_f)$ and plugging δS into (6) we get:

$$\frac{\partial}{\partial x_f} \langle x_f, t_f | x_i, t_i \rangle = \int D[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]} \frac{i}{\hbar} p(t_f) \tag{7}$$

We see that we obtain the value of the momentum at t_f by taking a partial derivative $\frac{\partial}{\partial x_f}$ of the propagator. Using our knowledge of Classical Mechanics again we obtain the Hamiltonian by varying the action with respect to t_f. Thus we have

$$\frac{\partial}{\partial t_f} \langle x_f, t_f | x_i, t_i \rangle = \int D[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]} \left(-\frac{i}{\hbar} H(t_f)\right)$$

If $H = \frac{\hat{p}^2}{2m} + V(\hat{x})$ we can rewrite the momentum using (7) to get

$$i\hbar \frac{\partial}{\partial t_f} \langle x_f, t_f | x_i, t_i \rangle = \left(\frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_f} \right)^2 + V(x_f) \right) \langle x_f, t_f | x_i, t_i \rangle$$

which we recognise as the well known Schrödinger Equation. Hence we have shown that the path integral formulation conveys the same information as non-relativistic quantum mechanics.

There is also a much easier, almost trivial, way to prove the equivalence. To do that we need to express the propagator using energy eigenstate expansion.

$$\begin{aligned} \mathcal{K} &= \langle x | \, e^{-\frac{i}{\hbar} H (t_f - t_i)} \, | x' \rangle \\ &= \sum_{n=1}^{\infty} \langle x | n \rangle \langle n | \, e^{-\frac{i}{\hbar} H (t_f - t_i)} \, | x' \rangle \\ &= \sum_{n=1}^{\infty} \phi_n(x) \, \phi_n^*(x') \, e^{-\frac{i}{\hbar} E_n (t_f - t_i)} \end{aligned}$$

where $\phi_n(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} px}$. Now we have to show that $\mathcal{K}$ satisfies the Schrödinger's equation:

$$\left[\frac{i}{\hbar} \frac{\partial}{\partial t} + \frac{1}{2m} \frac{\partial^2}{\partial x^2} \right] \mathcal{K} = 0$$

which is easily shown.

IV. FREE PARTICLE

Let's take a look at the simplest possible example - a free particle moving in one dimension. Let's discretise the path by dividing Δt into N pieces, so that the in-between points are $(x_1, t_1), \ldots, (x_{N-1}, t_{N-1})$. We identify $x_i = x_0$ and $x_f = x_N$. We do this for the chunks of the trajectory to form a continuous path in the $N \to \infty$ limit. As $V(\hat{x}) = 0$ the action depends just on the velocity of the particle. Let's find the action between t_n and t_{n+1}:

$$\begin{aligned} S_n = \int_{t_n}^{t_{n+1}} \frac{m}{2} \dot{x}(t) \, dt &= \frac{m}{2} \left(\frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right)^2 (t_{n+1} - t_n) \\ &= \frac{m}{2} \frac{(x_{n+1} - x_n)^2}{\Delta t} \end{aligned}$$

We will plug in the action as $S = \sum_{n=1}^{N} S_n$. Now to get the propagator we must vary all x_n at every t_n over the whole continuum.

$$\langle x_f, t_f | x_i, t_i \rangle = C(t) \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dx_1 \ldots dx_{N-1} \exp \left(\frac{i}{\hbar} \frac{m}{2} \frac{1}{N \Delta t} \sum_{n=1}^{N} (x_{n+1} - x_n)^2 \right) \quad (8)$$

where we denote the normalization constant as $C(t)$, which is dependent only on the elapsed time Δt. To integrate (8) that we will use the same trick as in the general case - we are going to integrate over just one of the variables. Let's conveniently choose to integrate over x_1

$$\int_{-\infty}^{\infty} \left(k \left(x_2^2 - x_1^2 \right) + k \left(x_1^2 - x_0^2 \right) \right) dx_1 = \sqrt{\frac{\pi}{2k}} e^{\frac{k}{2} (x_0 - x_2)^2}$$

where we've set $k = \frac{i}{\hbar} \frac{m}{2} \frac{1}{N \Delta t}$ for convenience. We also omitted some constant terms as they can be absorbed into $C(t)$. After integrating a few more times we see a pattern (which is shown in [4] explicitly): after n integrations a factor of $\frac{k}{n+1} (x_n - x_0)^2$ shows up in the exponential. We use this fact and recall that $N \Delta t = t_f - t_i$ to rewrite (8) as

$$\langle x_f, t_f | x_i, t_i \rangle = C(t_f - t_i) \, e^{\frac{k}{N} (x_f - x_i)^2}$$

So the propagator for a free particle for an elapsed time t is

$$\langle x, t | x_i, t_i \rangle = C(t) \, e^{\frac{im}{2t\hbar} (x - x_i)^2} \quad (9)$$

We obtain $C(t)$ by normalizing (9) over all x, keeping t constant. We get then

$$\langle x, t | x_i, t_i \rangle = \sqrt{\frac{m}{i 2\pi \hbar t}} e^{\frac{im}{2t\hbar} (x - x_i)^2} \quad (10)$$

We fix $C(t)$ to be the normalization constant for the path integral in general. Its rigorous derivation involving Green's functions can be found in chapter 13.3 of [7].

V. HARMONIC OSCILLATOR

Let us now take a look at path integral approach to a harmonic oscillator which is quite an elucidating example. We want to find the probability amplitude for a situation, where we leave a particle in the minimum of the harmonic potential at $x_i = 0$ and after measuring its position after a time $t = T$ we find it in the same place. Thus, we need to find a propagator

$$\begin{aligned} \mathcal{K}_{SHO} &= \langle x_f = 0, t_f = T | x_i = 0, t_i = 0 \rangle \\ &= \int_{x=0, t=0}^{x=0, t=T} D[x(t)] \exp \left(\frac{i}{\hbar} \int_{t=0}^{t=T} \left[\frac{m}{2} \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right] dt \right) \end{aligned}$$

Let us first take a look at the propagator for a harmonic oscillator in general.

We now need to find the classical trajectory. We start off by writing the general solution:

$$x(t) = A\cos\omega t + B\sin\omega t$$

To find A and B we set $x(t_i) = x_i$ and $x(t_f) = x_f$. We get then

$$\begin{cases} x_i & = A\cos\omega t_i + B\sin\omega t_i \\ x_f & = A\cos\omega t_f + B\sin\omega t_f \end{cases}$$

$$\begin{cases} A & = \frac{x_i - B\sin\omega t_i}{\cos\omega t_i} \\ B & = \frac{x_f\cos\omega t_i - x_i\cos\omega t_f}{\frac{1}{2}\sin(\omega(t_f - t_i))} \end{cases}$$

After some algebraic operations the classical path turns out to be

$$x_c(t) = x_i\frac{\sin(\omega(t_f - t))}{\sin(\omega(t_f - t_i))} + x_f\frac{\sin(\omega(t - t_i))}{\sin(\omega(t_f - t_i))}$$

The action integral then looks like

$$\begin{aligned} S_c &= \int_{t_i}^{t_f} dt\left(\frac{1}{2}m\dot{x_c}^2 - \frac{1}{2}m\omega^2 x_c^2\right) \\ &= \frac{1}{2}m\int_{t_i}^{t_f} dt\left(\dot{x_c}^2 - \omega^2 x_c^2\right) \\ &= \frac{1}{2}m\left(\int_{t_i}^{t_f} dt\dot{x_c}\dot{x_c} - \omega^2\int_{t_i}^{t_f} dt x_c x_c\right) \end{aligned}$$

We now integrate the first expression by parts:

$$S_c = \frac{1}{2}m\left(\dot{x_c}x_c\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt x_c\ddot{x_c} - \omega^2\int_{t_i}^{t_f} dt x_c x_c\right)$$

$\ddot{x_c}$ would normally bother us but we know that in the case of a simple harmonic oscillator $\ddot{x} = -\omega^2 x$. Using this we find the classical action to be

$$\begin{aligned} S_c &= \frac{1}{2}m\left(\dot{x_c}x_c\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt x_c\left(-\omega^2 x_c x_c\right) - \omega^2\int_{t_i}^{t_f} dt x_c x_c\right) \\ &= \frac{1}{2}m\dot{x_c}x_c\Big|_{t_i}^{t_f} \end{aligned}$$

$$S_c = \frac{1}{2}m\omega\frac{\left(x_i^2 + x_f^2\right)\cos(\omega(t_f - t)) - 2x_i x_f}{\sin(\omega(t_f - t_i))} \tag{11}$$

We can write $x(t) = x_c(t) + \delta x(t)$, where the quantum fluctuation $\delta x(t)$ must vanish at the initial and final time. We can thus expand $\delta x(t)$ in Fourier series around $x_c(t)$ following the derivation in [5] and get

$$S = S_c + \delta S = S_c + \sum_{n=1}^{\infty}\frac{m}{2}\left(\frac{(n\pi)^2}{t_f - t_i} - \omega^2(t_f - t_i)\right)\frac{a_n^2}{2}$$

We plug action into the propagator as an integral over all coefficients a_n:

$$\begin{aligned} \langle x_f, t_f | x_i, t_i\rangle = e^{\frac{i}{\hbar}S_c}\sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \\ \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\frac{n\pi}{\sqrt{2}}\int da_n \\ \exp\left[\frac{i}{\hbar}\sum_{n=1}^{\infty}\frac{m}{2}\left(\frac{(n\pi)^2}{t_f - t_i} - \omega^2(t_f - t_i)\right)\frac{a_n^2}{2}\right] \end{aligned}$$

Where we used the normalization constant fixed in the previous section.
After integrating we get

$$\begin{aligned} \langle x_f, t_f | x_i, t_i\rangle = e^{\frac{i}{\hbar}S_c}\sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \\ \prod_{n=1}^{\infty}\left(1 - \left(\frac{\omega(t_f - t_i)}{n\pi}\right)^2\right)^{-\frac{1}{2}} \end{aligned}$$

We recognize the latter term as an infinite product representation of a sinc function:

$$\prod_{n=1}^{\infty}\left(1 - \frac{\frac{\omega^2}{\pi^2}(t_f - t_i)^2}{n^2}\right) = \frac{\sin(\omega(t_f - t_i))}{\omega(t_f - t_i)}$$

So we find the propagator to be

$$\begin{aligned} \langle x_f, t_f | x_i, t_i\rangle &= e^{\frac{i}{\hbar}S_c}\sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\sqrt{\frac{\omega(t_f - t_i)}{\sin(\omega(t_f - t_i))}} \\ &= \sqrt{\frac{m}{2\pi i\hbar}}e^{\frac{i}{\hbar}S_c}\sqrt{\frac{\omega}{\sin(\omega(t_f - t_i))}} \end{aligned} \tag{12}$$

Now, having calculated the propagator, let us revisit the situation from the beginning of this section. We realise that $S_c = 0$ for this case, so the propagator will be

$$\langle x_f = 0, t_f = T | x_i = 0, t_i = 0\rangle = \sqrt{\frac{\omega}{\sin\omega T}}$$

Now we know the probability to find a particle left in the minimum of the harmonic potential in the same place after a time T!

Note that after we take the limit of $\omega \to 0$ of $\mathcal{K}_{SHO}$ we end up with

$$\begin{aligned} \lim_{\omega\to 0}\mathcal{K}_{SHO} = \\ \sqrt{\frac{m}{2\pi i\hbar}}e^{\frac{i}{\hbar}S_c}\lim_{\omega\to 0}\sqrt{\frac{\omega(t_f - t_i)}{\sin(\omega(t_f - t_i))}}\sqrt{\frac{1}{(t_f - t_i)}} \\ = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}e^{\frac{i}{\hbar}S_c} \end{aligned}$$

which is, as we would expect, the propagator for a free particle!

VI. IMAGINARY TIME

We should note that the path integral is not a rigorously defined object. The issue occurs when we try to deal with paths rapidly oscillating around the classical path. Such rapid oscillations can cause convergence issues. They are in general not pleasant to deal with unless we define imaginary time $\tau = it$. This step, known as Wick rotation, is not easy to justify so it will not be covered here. It is based on some characteristics shared by both Minkowski measure and Euclidean measure. An interested reader may refer to section 11.5 of [7] or section 14 of [8]. Let us take a look at a propagator over a small time interval $-i\tau$

$$\mathcal{K}_{\text{im}} \approx \sum_{\text{all paths}} e^{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x'-x)^2}{-i\tau}+(-i\tau)V\left(\frac{x+x'}{2}\right)\right]}$$

We can see that the potential has changed its sign relative to the kinetic energy term. Let's then define the Euclidean action as

$$S_E\left[x\left(\tau\right)\right] = \int_{\tau_i}^{\tau_f} d\tau \left[\frac{m}{2}\dot{x}^2\left(\tau\right) + V\left(x\left(\tau\right)\right)\right]$$

We can then define the imaginary time propagator

$$\mathcal{K}_{\text{im}} = C\left(\tau\right)\int_{\text{all paths}} \exp\left(-\frac{1}{\hbar}S_E\left[x\left(\tau\right)\right]\right)$$

Instead of causing oscillation, deviations from the classical path contribute as an average weighted by their Euclidean action! The classical path will dominate, as action along it is minimized.

Let us now write a propagator in imaginary time using energy eigenstate expansion:

$$\begin{aligned}\mathcal{K}_{\text{im}} &= \langle x'|\, e^{-\frac{i}{\hbar}H(-i\tau)}\,|x\rangle \\ &= \sum_n \langle x'|n\rangle \langle n|\, e^{-\frac{1}{\hbar}H\tau}\,|x\rangle \\ &= \sum_n \langle x'|n\rangle\, e^{-\frac{1}{\hbar}E_n\tau}\,\langle n|x\rangle\end{aligned}$$

Now if we set $x = x'$ and integrate over all x

$$\begin{aligned}\int_{-\infty}^{\infty} dx\mathcal{K}_{\text{im}} &= \sum_n \langle n|\int_{-\infty}^{\infty} dx e^{-\frac{1}{\hbar}E_n\tau}\,|x\rangle\langle x|n\rangle \\ &= \sum_n e^{-\frac{\tau}{\hbar}E_n} \qquad (13)\end{aligned}$$

Note that this result resembles the partition function known from statistical mechanics $Z = \sum_n e^{-\beta E_n}$, where $\beta = \frac{1}{k_b T}$.

Let us now consider the imaginary time propagator for a simple harmonic oscillator:

$$\begin{aligned}\mathcal{K}_{\text{im SHO}} &= \langle x'|\, e^{-\frac{1}{\hbar}H\tau}\,|x\rangle \\ &= \sum_n \langle x'|n\rangle\langle n|\, e^{-\omega\left(n+\frac{1}{2}\right)\tau}\,|x\rangle \\ &= \sum_n \phi_n\left(x\right)\phi_n^*\left(x'\right) e^{-\omega\left(n+\frac{1}{2}\right)\tau}\end{aligned}$$

Now if we redo the step from (13) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} dx\mathcal{K}_{\text{im SHO}} &= \sum_n e^{-\omega\left(n+\frac{1}{2}\right)\tau} \\ &= e^{-\tau\frac{\omega}{2}}\sum_n e^{-(\tau\omega)^n} \\ &= e^{-\tau\frac{\omega}{2}}\frac{1}{1-e^{-\tau\omega}} \\ &= \frac{1}{2\sinh\left(\frac{\omega\tau}{2}\right)}\end{aligned}$$

Note that if we use $\tau = it$ we get

$$\int_{-\infty}^{\infty} dx\mathcal{K}_{\text{im SHO},\tau\to it} = \frac{1}{2i\sin\left(\frac{\omega t}{2}\right)} \qquad (14)$$

We check that this is true by solving the left-hand side of above expression. We recall (11) and rewrite it to fit our goal, setting $x_i = x_f = x$:

$$\begin{aligned}S_C &= \frac{1}{2m}\frac{2x^2\cos\left(\omega\left(t_f-t\right)\right)-2x^2}{\sin\left(\omega\left(t_f-t_i\right)\right)} \\ &= \frac{m\omega\left[\cos\left(\omega\left(t_f-t\right)\right)-1\right]x^2}{\sin\left(\omega\left(t_f-t_i\right)\right)} \\ &= \frac{-2m\omega\sin^2\left(\frac{\omega(t_f-t)}{2}\right)x^2}{\sin\left(\omega\left(t_f-t_i\right)\right)}\end{aligned}$$

Where we used the identity $\cos x - 1 = -2\sin^2\frac{x}{2}$. We now plug this into $\mathcal{K}_{\text{SHO}}$.

$$\begin{aligned}\int_{-\infty}^{\infty} dx\mathcal{K}_{\text{SHO}} &= \sqrt{\frac{m}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\left(\omega\left(t_f-t_i\right)\right)}}\int_{-\infty}^{\infty} dx e^{\frac{i}{\hbar}S_C} \\ &= \sqrt{\frac{m}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\left(\omega\left(t_f-t_i\right)\right)}} \\ &\quad\times\sqrt{\frac{\pi i\hbar\sin\left(\omega\left(t_f-t_i\right)\right)}{-2m\omega\sin^2\left(\frac{\omega(t_f-t)}{2}\right)}} \\ &= \frac{1}{2}\sqrt{\frac{1}{-\sin^2\left(\frac{\omega(t_f-t)}{2}\right)}} \\ &= \frac{1}{2i\sin\left(\frac{\omega(t_f-t)}{2}\right)}\end{aligned}$$

which is exactly the same as (14)!

VII. CONCLUSION

In this paper we have introduced the path integral formulation of quantum mechanics. There is a lot of literature richer in mathematical rigour, such as [5] or [6], while other resources put more focus on the physical intuition as [3]. We tried to keep balance between both approaches.

One of the most important drawbacks of this formulation is the lack of mathematical rigour. Mathematicians consider the path integral not to be a rigorously defined object. Even though the discussed method is often described as beautiful many find it too technical and computation-heavy.

Feynman hoped this approach would help replace QED with particle quantum mechanics. Even though this method failed to serve its original purpose, it has proven itself very useful as a different way to obtain classical results in quantum mechanics. It has also turned out to be very successful in quantum field theory as a second formulation of the theory, along canonical quantization.

We have derived the general case propagator in a way that illuminates the meaning behind summing over all possible paths. We have shown the equivalence of this method to the Schrödinger formulation. We have also applied the path integral to a free particle, using this simple example to fix the normalization of the integral. Then we derived the probability amplitude to find a particle in the minimum of the harmonic potential after a time T.

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