HW6 due 05/02/2022

(+2 points for handing in on time)

Q.1 Matsubara sum.

a) Study (again!) the basic integral

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2 + \omega^2} = \frac{1}{2\omega}.$$

Consider

$$p_4 = n\Delta p_4$$
$$\Delta p_4 = \frac{2\pi}{L_4},$$

where $n = -N_{\max}, -N_{\max}+1, \ldots, N_{\max}$ and convert the integral into a Riemann sum

$$\frac{1}{L_4} \sum_{n=-\infty}^{\infty} \frac{1}{(\Delta p_4)^2 \; n^2 + \omega^2}.$$

Numerically evaluate the sum (for large enough $N_{\max}$ and small enough Δp_4) and verify the integral.

b) The case of a finite L_4 is an important result in finite temperature field theory: $L_4 \to \beta = 1/T$ is the inverse temperature, $p_4 \to \omega_n = \frac{2\pi}{\beta} n$ are the Matsubara frequencies (for Bosons), and the sum of interest reads

$$G(\beta, \omega) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \omega^2}.$$

The analytic result is

$$G(\beta, \omega) = \frac{1}{2\omega} \coth(\frac{\beta\omega}{2}).$$

To derive this result, we can use the Euler's product formula for $\sin(x)$

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Make sense of the sine formula by numerically computing the RHS (for a large $N_{\max}$) and plotting the two functions.

c) Show that

$$\sin(ix) = i\sinh(x) = i\frac{e^x - e^{-x}}{2}$$

and obtain an analogous product formula for $\sinh(x)$. The result is

$$\sinh(x) = x\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2\pi^2}\right).$$

d) Take the logarithm on both sides and reach

$$\ln\sinh(x) = \ln x + \sum_{n=1}^{\infty}\ln\left(1 + \frac{x^2}{n^2\pi^2}\right).$$

Now take the derivative (with respect to x) on both sides and show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2\pi^2 + x^2} = \frac{1}{x}\coth(x).$$

e) Finish the job to show that

$$G(\beta,\omega) = \frac{1}{2\omega}\coth(\frac{\beta\omega}{2}).$$

What is the $\beta \to \infty$ limit? Explain.

Q.2

In the (3D) Fourier transform of $\frac{1}{k^2+m^2}$ to the x-space we encounter the integral

$$\begin{aligned}&\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\,\frac{1}{k^2+m^2}\\ &= \frac{1}{2\pi^2 r}\int_0^{\infty}dk\,k\,\sin(kr)\,\frac{1}{k^2+m^2}.\end{aligned}$$

The integral can be difficult to handle numerically due to the oscillatory behavior at large k. Consider instead a regulated version of the integral, e.g.

$$I(r;\Lambda) = \frac{1}{2\pi^2 r}\int_0^{\infty}dk\,k\,\sin(kr)\,\frac{1}{k^2+m^2}\,e^{-k^2/\Lambda^2}.$$

a) Evaluate the regulated integral at several Λ's.

b) Show that the numerical result becomes stable at large values of Λ's, and approaches the analytic limit (remember what the answer is?).

Q.3

a) From $S = e^{2i\delta} = 1 + it$, show that $2\text{Im}t = |t|^2$.

b) Given the scattering amplitude for a resonant process can be parametrized as

$$f(E) \propto \frac{1}{E - E_R + i\frac{\gamma}{2}},$$

show that the resonant phase shift $\delta_R(E)$ satisfies

$$\tan \delta_R(E) = \frac{-\gamma/2}{E - E_R}.$$

Plot the phase shift function with some reasonable values of the parameters.

Q.4

Derive the key results in the scattering theory:

a) The Green's function: show that

$$\begin{aligned} G_E^0(x) &= \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \frac{1}{E - \frac{q^2}{2m_R} + i\delta} \\ &= \frac{1}{2\pi^2 r} \int_0^\infty dq\, \sin(qr)\, \frac{q}{E - \frac{q^2}{2m_R} + i\delta} \\ &= -2m_R \times \frac{1}{4\pi r}\, e^{+ipr}. \end{aligned}$$

b) Show that the G_E^0 satisfies

$$(E - \hat{H}_0) G_E^0(\vec{x_1}, \vec{x_2}) = \delta^{(3)}(\vec{x_1} - \vec{x_2})$$

where $\hat{H}_0 = -\frac{\nabla^2}{2m_R}$.

c) The full Green's function G_E satisfies

$$(E - \hat{H}) G_E(\vec{x_1}, \vec{x_2}) = \delta^{(3)}(\vec{x_1} - \vec{x_2}).$$

where $\hat{H} = \hat{H}_0 + \hat{V}$. Derive the relation between G_E and G_E^0.

d) What about the full wavefunction ψ and the scattering amplitude f? How can they be extracted from G^0?

Q.5

a) Numerically compute the S-wave phase shift $\delta(E)$ for a finite barrier:

$$V(r) = V_0,$$

if $x < R$, otherwise zero. You can take $m_R = 1$, $V_0 = 2.0$, $R = 1.5$. Plot the result in a suitable energy range: e.g. 0.1:14.0.

b) The effective spectral function

$$\Delta A(E) = 2\,\frac{d}{dE}\delta(E)$$

corresponds to the change of the density of states due to interaction. Compute $\Delta A(E)$ for the finite barrier problem.