NumQM Spring2022

HW1 due 20/03/2022

(+1 points if you solve the problems with Julia)

(+1 points for handing in on time)

Q1 Residue theorem.

The Cauchy's residue theorem states that the line integral of an analytic function over a closed path Γ is given by ($2\pi i$ times) the sum of residues:

$$\oint_\Gamma dz f(x) = 2\pi i \sum_{\text{Res.}} f(z_n).$$

- Take

  $$f(z) = \frac{1}{z^2+1}$$

  and verify the residue theorem by numerically computing the integral along the paths:

  - A) a square centered around the pole $z = i$;
  - B) an upper semi-circle of radius $R = 2$. Separate the contributions from the real line: -R:R and the arc.

- Repeat the calculation in B) for a larger R. Show that the arc contribution decreases as $R \to \infty$ and only the real-line contribution remains, hence

  $$\int_{-\infty}^{\infty} dx \, \frac{1}{x^2+1} = \pi.$$

Q2 Schwinger proper time regularization.

Consider the following identities:

$$\mathcal{A}^{-1} = \int_0^\infty dt \, e^{-t\mathcal{A}}$$

$$\ln \mathcal{A} = -\int_0^\infty dt \, \frac{1}{t}\left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}}\right).$$

- With $\mathcal{A} \to p_4^2 + \omega^2$, show (again!) that

  $$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2+\omega^2} = \frac{1}{2\sqrt{\omega^2}}.$$

- The second identity is useful for regulating divergent integral. Consider

$$\begin{aligned} W[\omega;\Lambda] &= \int_{-\infty}^{\infty} \frac{dp_4}{2\pi}\, \ln\left(p_4^2+\omega^2\right) \\ &\to \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int_{1/\Lambda^2}^{\infty} dt\, (-\frac{1}{t})\, e^{-t(p_4^2+\omega^2)} + C \\ &= -\int_{1/\Lambda^2}^{\infty} dt\, \frac{1}{t^{\frac{3}{2}}}\, \frac{1}{2\sqrt{\pi}}\, e^{-t\omega^2} + C. \end{aligned}$$

  Compute $W[\omega;\Lambda]$ numerically for a large enough Λ. You can forget about the integration constant C. Compare with the analytic result:

$$W[\omega;\Lambda] = -\frac{\Lambda}{\sqrt{\pi}} + \omega + \mathcal{O}(1/\Lambda).$$

- This suggests the definition of a physical W function:

$$W_{\text{phys.}}[\omega] = \lim_{\Lambda\to\infty} \left(W[\omega;\Lambda] - W[0;\Lambda]\right) = \omega.$$

  Re-derive the previous result via

$$\int_{-\infty}^{\infty} dx\, \frac{1}{x^2+\omega^2} = \frac{1}{2\omega}\frac{\partial}{\partial\omega} W_{\text{phys.}}[\omega].$$

- Generate a 3 x 3 positive definite matrix $\mathcal{A}$. Compute the inverse and determinant using the above identities.

Q3 Kinematics and threshold.

- Consider the scattering of two particles (of mass m_1 and m_2) in the Center-of-Mass (COM) frame. The energy in this frame is also the invariant mass $\sqrt{s}$ of the system, i.e.

$$E_{\text{COM}} = \sqrt{s} = \sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2}.$$

  Show that

$$q(s) = \frac{1}{2}\sqrt{s}\sqrt{1-\frac{(m_1+m_2)^2}{s}}\sqrt{1-\frac{(m_1-m_2)^2}{s}}.$$

- Explain why

$$\text{Im}\left(\ln(-1\pm i\delta)\right) \to \pm\pi$$

  where δ is a small (positive) number. Numerically verify these results.

- A QFT computation of the self energy Σ_R of a resonance gives

$$\Sigma_R(s) = 2 \times \frac{g^2}{16\pi^2} \int_0^1 dx \ln\left(m^2 - x(1-x)s - i\delta\right).$$

(The factor of 2 is the symmetry factor for identical particles.) Show that the self energy develops an imaginary part when the COM energy is above the threshold:

$$\text{Im}\Sigma_R(s) = -2 \times \frac{1}{2} g^2 \frac{q}{4\pi\sqrt{s}} \theta(\sqrt{s} - 2m)$$

where

$$q = \frac{1}{2}\sqrt{s}\sqrt{1 - \frac{4m^2}{s}}$$

corresponds to the COM momentum for the case of $m_1 = m_2 = m$.

Q4 Logistic Map (again).

Consider the logistic map:

$$x_{n+1} = r x_n (1 - x_n).$$

- Produce the bifurcation diagram: (see Sem1 HW01 Q5) for a given r within $(0, 4)$, iterate (starting from any initial value within $(0, 1)$), collect the end point(s) and plot them as a function of r.
- Derive an analytic result for the stable end points within $r < 3$. Plot it on top of the bifurcation diagram.
- For $r = 3.6$, collect all the points $\{x_j\}$ from iterations and plot them in a normalized histogram. This is called the invariant density.
- Plot the invariant density at $r \to 4$. Compare with the analytic result:

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

- Here's how to understand the analytic result. Consider

$$f(x) = 4x(1-x).$$

Note that $f(1-x) = f(x)$. Find the two $x_j(x)$ $(j = 1, 2)$ that are mapped to x, i.e.

$$x = f(x_j).$$

- The invariant density $\rho(x)$ can be written as

$$\rho(x) = \int_0^1 dx' \, \rho(x') \, \delta(x - f(x'))$$
$$= \sum_{j=1,2} \frac{\rho(x_j)}{|f'(x_j)|}$$

  where $f'(x_j)$ is the derivative of f evaluated at x_j. The task becomes verifying the functional form

$$\rho(x) = \frac{K}{\sqrt{x(1-x)}}$$

  works. Finally, fix the constant $K = 1/\pi$ by requiring

$$\int_0^1 \rho(x) = 1.$$

Q5 Particle adventure.

Visit the PDG website link to look up particle properties.

- Look up the Fermi coupling constant G_F and mass of W-boson M_W.
- Given

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8M_W^2},$$

  what is the value of the weak interaction coupling constant g_W and $\alpha_W = \frac{g_W^2}{4\pi}$? Compare to the QED value

$$\alpha_{EM} = \frac{1}{137},$$

  which one is stronger?

- How many pions are there? And how about the kaons? Include anti-particles and spin degeneracies in the counting, e.g. N(electron) = 4: electron (spin up, spin down); positron (spin up, spin down)
- Perform a similar counting for the $\Delta(1232)$ resonance.