NumQM Spring2022

HW2 due 10/04/2022

(+1 points if you solve the problems with Julia)

(+1 points for handing in on time)

Q1 Low and High Temperature expansions.

- Fill in the steps in deriving the exact solution of the 1D Ising Model based on Transfer Matrix method. Obtain the analytic expressions of the partition function and average spin.
- Compare the results to the corresponding low and high temperature expansions of the partition function.
- Derive the low and high temperature expansions of the energy per site for the d-dimensional Ising Model (zero external field).

Q2 2D Ising Model.

- Write down the partition function in the low and high temperature expansions (at least 2 correction terms in each case).
- Compare the correction factors to the corresponding leading term. Show that the two series are the same if we map

  $$e^{-2K} \leftrightarrow \tanh(K).$$

  Plot the two functions versus K.
- Obtain an analytic expression of the critical coupling K_c at the point of intersection. (K is just the inverse temperature for $J = 1$)
- The exact solution, due to Onsager, for the partition function reads (see textbook by Pathria or Kerson Huang)

  $$\ln Z_{\text{persite}} = \ln\left(\sqrt{2}\cosh 2K\right) + \frac{1}{\pi}\int_0^{\pi/2} d\phi \ln\left(1 + \sqrt{1 - \kappa^2 \sin^2\phi}\right)$$
  $$\kappa = \frac{2\sinh 2K}{\cosh^2 2K}.$$

  Plot the exact solution for energy per site versus temperature. Compare with the low and high temperature expansions.

Q3 Feynman Parametrization.

- Prove the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2}.$$

- The self energy Σ_R of a resonance is obtained from

$$\Sigma_R = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q_1^2 - m_1^2 + i\delta} \frac{1}{q_2^2 - m_2^2 + i\delta}$$

  where

$$\begin{aligned} q_1 &= q \\ q_2 &= P - q \\ P^2 &= s = (P^0)^2 - \vec{P}^2 \end{aligned}$$

  are Minkowski 4-vectors. Use the Feynman parametrization to show that

$$\Sigma_R = ig^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{((\tilde{q})^2 - \Delta)^2}$$

  where

$$\begin{aligned} \tilde{q} &= q - (1-x)P \\ \Delta &= xm_1^2 + (1-x)m_2^2 - x(1-x)s - i\delta. \end{aligned}$$

- After a shift of integration variable and a Wick's rotation:

$$\begin{aligned} d^4q &\to i d^4q_E \\ q^2 &\to -q_E^2 = -(q_4^2 + \vec{q}^2), \end{aligned}$$

  we obtain

$$\Sigma_R = -g^2 \int_0^1 dx \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{(q_E^2 + \Delta)^2}$$

  where q_E is in Euclidean space.

  Use the Schwinger proper time regularization and perform the momentum integral. Show that

$$\Sigma_R = \frac{g^2}{16\pi^2} \int_0^1 dx \ln \Delta + C.$$

  This is the starting point of Q3 in HW01.

  Hint: Recall the Schwinger proper time regularization scheme

$$
\begin{aligned}
\mathcal{A}^{-1} &= \int_0^\infty dt\, e^{-t\mathcal{A}} \\
\ln \mathcal{A} &= -\int_0^\infty dt\, \frac{1}{t} \left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}} \right).
\end{aligned}
$$

The following relation is also useful:

$$
\mathcal{A}^{-2} = \int_0^\infty dt\, t\, e^{-t\mathcal{A}}.
$$

Q4 Numerical study of 4D Ising Model (x2 points)

- Write a program to solve the 4D Ising Model. What is the critical temperature you get?
- Plot the average spin and the spin susceptibility as functions of temperature.
- Compute the average energy per site. Compare to the low and high temperature expansions.