NumQM Spring2022

HW3 due 01/05/2022

(+1 points if you solve the problems with Julia)

(+1 points for handing in on time)

Q1 Dual of 3D Ising model

a) Show that the low temperature expansion of the 3D Ising model is equivalent to the high temperature expansion of a 3D Z(2) gauge theory. (work out the first 2 correction terms)

b) What about the high temperature expansion of the 3D Ising model? Is it equivalent to the low temperature expansion of the Z(2) gauge theory?

c) Argue that the the dual of 4D Z(2) gauge theory is another Z(2) gauge theory.

Q2

a) Show that the heat capacity

$$c = \frac{\partial}{\partial T}\epsilon$$

where $\epsilon = \langle E \rangle / N$, can be obtained from measuring the fluctuation of the total energy, i.e.

$$c = \beta^2 \frac{1}{N}\left(\langle E^2 \rangle - \langle E \rangle^2\right).$$

b) Numerically realize the equivalence of the two for Ising Model in 4D.

Q3

a) Consider

$$G(\lambda) = e^{\lambda H} A e^{-\lambda H},$$

derive the differential equation

$$\frac{d}{d\lambda} G(\lambda) = [H, G(\lambda)].$$

b) Show that

$$G(1) = A + [H, A] + \frac{1}{2!}[H, [H, A]] + \ldots.$$

c) Choose two 3×3 matrices A and H and use the Euler method (1st order) to solve for $G(\lambda = 1)$ numerically. Compare with a direct computation of $G(1)$.

d) Prove the Baker-Campbell-Hausdorff formula:

$$e^A\, e^B = e^C$$
$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}\left([A, [A, B]] - [B, [A, B]]\right) + \ldots.$$

Q4

The N-body Lorentz Invariant phase space (LISP) is defined as

$$\phi_N(s = P^2) = \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\cdots\frac{d^3p_N}{(2\pi)^3}\frac{1}{2E_N}\times$$
$$(2\pi)^4\,\delta^4(P - \sum_i p_i).$$

Note that $E_j = \sqrt{p_j^2 + m_j^2}$.

a) Show that the $N = 2$-body phase space is given by

$$\phi_2(s) = \frac{q(s)}{4\pi\sqrt{s}}$$
$$q(s) = \frac{1}{2}\sqrt{s}\sqrt{1 - \frac{(m_1 + m_2)^2}{s}}\sqrt{1 - \frac{(m_1 - m_2)^2}{s}}.$$

b) Numerically compute the 2-body phase space integral and check with the analytic result. Plot the results as a function of $\sqrt{s}$.

Q5

$W = \ln Z$ is extensive.

Introducing an external field h to the integral

$$Z(h) = \int_{-\infty}^{\infty} dx\, e^{-V(x^2+x^4)+V h\, x},$$

and define

$$Z_n = \frac{1}{Z(h)}\,\partial_h^n Z(h) = \langle x^n \rangle \times V^n$$

and

$$W(h) = \ln Z(h)$$
$$W_n = \partial_h^n W(h).$$

a) Show that (see Sem01 HW02 Q4, but this time we do not set $h \to 0$ at the end.)

$$\begin{aligned}
W_1 &= Z_1 \\
W_2 &= Z_2 - Z_1^2 \\
W_3 &= Z_3 - 3Z_1Z_2 + 2Z_1^3 \\
W_4 &= Z_4 - 4Z_1Z_3 - 3Z_2^2 + 12Z_1^2Z_2 - 6Z_1^4
\end{aligned}$$

b) Verify these relations numerically, i.e. taking numerical derivatives on $W(h)$ and performing corresponding numerical integrations

$$Z_n = \frac{1}{Z(h)} \int_{-\infty}^{\infty} dx\, V^n x^n\, e^{-V(x^2+x^4)+Vh\,x}.$$

Plot them as functions of h in the range of -2:2. (Take V = 2.)

c) Verify the linked cluster theorem: $W_n \propto V$ at large V, i.e. $W_n/V = w_n$ becomes intensive.