NumQM Spring2022

HW4 due 21/05/2022

(+1 points if you solve the problems with Julia)

(+1 points for handing in on time)

Q1 Spectral function of free particle

a) Consider the propagator of free particle

$$G(x, x', t) = \sqrt{\frac{m}{2\pi i t}}\, e^{i\frac{1}{2}m\frac{(x-x')^2}{t}}.$$

Show that its Fourier transform to the energy space is given by

$$\begin{aligned}\tilde{G}_R(x = x', E) &= -i\int_0^\infty dt\, e^{iEt} G(x = x', t)\\ &= -i\frac{m}{p},\end{aligned}$$

where $E = \frac{p^2}{2m}$.

b) Consider the eigenstate expansion for $\tilde{G}_R$:

$$\tilde{G}_R(x = x', E) = \int \frac{dp}{2\pi}\, \frac{1}{E - \frac{p^2}{2m} + i\delta}.$$

Compute the spectral density

$$\rho(E) = -2\mathrm{Im}\tilde{G}_R(x = x', E).$$

Verify this with the result in part a).

Q2 Numerical Path Integral

a) Write (or reuse) a numerical program to solve the 1D Schroedinger Equation, with potential

$$V(x) = \frac{1}{2}x^4.$$

You can take m = 1. Find the wavefunctions and energies of the lowest 4 states.

b) Set up a numerical path integral for the same problem. Extract the ground state wavefunction. Compare this with the one obtained in part a).

c) Compute $\langle x^4 \rangle$ (e.g. based on Metropolis) within the path integral approach. Use the virial theorem to relate this to the ground state energy. Verify the result numerically.

d) Explain what is wrong in computing the energy, i.e. expectation value of the Hamiltonian, via

$$\left\langle \frac{1}{2} m \frac{(x_{j+1} - x_j)^2}{(\delta\tau)^2} + V(x_j) \right\rangle$$

.

What would be the correct way?

Q3 The Dalitz plot.

a) Show that the invariant mass ($\sqrt{s}$) of a 3-body system satisfies

$$s = s_{12} + s_{23} + s_{13} - m_1^2 - m_2^2 - m_3^2,$$

where standard relativistic kinematics apply, i.e.

$$\begin{aligned} s_{ij} &= (p_i + p_j)^2 \\ p_i &= (E_i, \vec{p}_i) \\ p_i^2 &= E_i^2 - (\vec{p}_i)^2. \end{aligned}$$

b) Construct a Dalitz plot for the system of 3 pions in the following manner:

- Given s, randomly generate the 3-momenta of the 3 pions, and construct [s12, s23, s13].
- Collect all the events which satisfy the constraint in part a) (up to around 5000 event)
- plot a scatter plot for [s12, s23] in the collection. (you can choose any 2)

You can take $s = 0.65$ GeV^2

c) Interpret the various boundaries of the plot. See PDG Fig. 47.3

Q4 3 ways to 3-body phase space.

The N-body Lorentz Invariant phase space (LISP) is defined as

$$\phi_N(s = P^2) = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} \cdots \frac{d^3p_N}{(2\pi)^3} \frac{1}{2E_N} \times$$
$$(2\pi)^4 \, \delta^4(P - \sum_i p_i).$$

Note that $E_j = \sqrt{p_j^2 + m_j^2}$.

a) Numerically compute the $N = 3$-body phase space. Take all masses to be the pion mass $m = 0.14$ GeV. Choose a reasonable invariant mass sqrt(s).

b) Show that the same result can be computed via

$$\phi_3(s) = \frac{1}{32\pi^3} \frac{1}{4s} \int ds_{12} \int ds_{23},$$

where s_{12} ranges from $(m_1 + m_2)^2$ to $(\sqrt{s} - m_3)^2$. Can you figure out the range of s_{23}? Note that it depends on s_{12} and this integral needs to be done first. (see the PDG notes for hints.)

c) Verify the third method to compute the 3-body phase space, using the recursion formula

$$\phi_3(s, m_1^2, m_2^2, m_3^2) = \int_{s'_-}^{s'_+} \frac{ds'}{2\pi} \, \phi_2(s, s', m_3^2) \; \phi_2(s', m_1^2, m_2^2),$$

where

$$s'_- = (m_1 + m_2)^2$$
$$s'_+ = (\sqrt{s} - m_3)^2,$$

and the 2-body phase space $\phi_2(s, m_1^2, m_2^2)$ is given by (remember?! HW03 Q4)

$$\phi_2(s, m_1^2, m_2^2) = \frac{q(s)}{4\pi\sqrt{s}}$$
$$q(s) = \frac{1}{2}\sqrt{s}\sqrt{1 - \frac{(m_1 + m_2)^2}{s}}\sqrt{1 - \frac{(m_1 - m_2)^2}{s}}.$$

Q5 Free Gas.

Review the computation of thermal pressure for free gas of bosons and fermions. Take $\mu_B = 0$.

a) Compute the thermal pressure for a gas of free pions (P_1). Plot P/T^4 as a function of T.

b) Compute the thermal pressure for a free gas of quarks (u and d) and gluons. (P_2). Work out their degeneracies. Plot P/T^4 on the same plot as part a).

c) Find the bag constant B such that

$$P_1(T_c) = P_2(T_c) - B$$

at $T_c = 0.16$ GeV.

d) Numerically construct a model equation of states such that:

P_model(T) = P_1(T) for T < Tc

P_model(T) = P_2(T) - B for T >= Tc

Show that P_model(T) is continuous at Tc, but the entropy density exhibits a jump. Plot the results.

e) Extract the latent heat L. The expected result is $L \approx 4 \times B$.