NumQM Spring2022

HW5 due 10/06/2022

(+1 points if you solve the problems with Julia)

(+1 points for handing in on time)

Q1 Debye Screening in an electron gas

a) The two-body interaction among electrons, i.e. the (static) Coulomb potential, is dressed by the ring diagram defined as

$$\Pi(k^0, \vec{k}) = 2_{\text{spin}}\, e^2 \int \frac{d^3\ell}{(2\pi)^3} \frac{n(E_1) - n(E_2)}{k^0 + E_1 - E_2 + i\delta}$$
$$E_1 = \frac{\vec{\ell}^2}{2m}$$
$$E_2 = \frac{(\vec{\ell} + \vec{k})^2}{2m}.$$

with the occupation number $n(E)$ approximated by the low temperature limit (large β) of

$$n(E) \to \frac{1}{e^{\beta(E-E_F)} + 1},$$

$E_F = \frac{k_F^2}{2m}$ defines the Fermi energy.

Study the static limit of the ring diagram: $k^0 = 0$, and derive the analytic result

$$m_D^2 = -\Pi(k^0 = 0, \vec{k} \to \vec{0}) = e^2\, \frac{mk_F}{\pi^2},$$

where m_D is called the Debye mass.

b) Numerically study the ring diagram at $k^0 = 0$ but for general k. Plot this as a function of k. Compare with the analytic results at small and large k:

$$\Pi^{\text{static}}(k^0 = 0, k \to 0) \approx -m_D^2 \left(1 - \frac{1}{12}\frac{\vec{k}^2}{k_F^2}\right)$$

$$\Pi^{\text{static}}(k^0 = 0, k \to \infty) \to -m_D^2 \frac{4}{3}\frac{k_F^2}{k^2}.$$

c) The ring diagram dresses the potential via

$$\begin{aligned}
V(\vec{k}) = e^2 D_0 &\longrightarrow \tilde{V}(k^0,\vec{k}) = e^2 \tilde{D}_0 \\
\tilde{D}_0(k^0,\vec{k}) &= \frac{1}{D_0^{-1} - \Pi(k^0,\vec{k})} \\
D_0(\vec{k}) &= \frac{1}{\vec{k}^2}.
\end{aligned}$$

When we retain only the Debye screening mass, the effective potential becomes

$$\tilde{V}(k^0 = 0,\vec{k}) \approx \frac{e^2}{\vec{k}^2 + m_D^2}.$$

Show that the x-space potential is effectively screened:

$$\frac{e^2}{4\pi r} \to \frac{e^2}{4\pi r}\, e^{-m_D r}.$$

Verify this numerically.

d) The ring contribution to the partition function reads

$$\begin{aligned}
\Delta \ln Z_{\text{ring}} &= -\frac{1}{2}\,\text{tr}\left(\ln(1 - D_0\Pi) + D_0\Pi\right) \\
&\approx -\frac{1}{2} V \int \frac{d^3k}{(2\pi)^3} \left(\ln(1 + \frac{m_D^2}{k^2}) - \frac{m_D^2}{k^2} \right).
\end{aligned}$$

The second line follows from leading order static approximation. Compute the integral and show that it leads to a non-analytic contribution (in powers of e^2) to the thermal pressure.

e) Now restore the k^0 dependence. The collective propagation mode $\omega(k)$ can be identified from the equation:

$$D_0(\vec{k})^{-1} - \Pi(\omega(k),\vec{k}) = 0.$$

Numerically solve for $\omega(k)$ (you may assume it to be real) for $k < 0.3k_F$. This is called the plasmon mode. (Take $k_F = m = 1$ for simplicity.)

Hint: For each k, solve for the ω from the common root of the real and imaginary part of the equation.

Write your result in terms of

$$\omega(k)^2 \approx \omega_D^2 + \alpha k^2.$$

Compare with the analytic results:

$$\omega_D^2 = \frac{1}{3}\, m_D^2$$
$$\alpha = \frac{3}{5}.$$

Q2 Cluster / Virial Expansion

a) Consider the cluster expansion of the thermodynamic pressure

$$\frac{P}{n_0 T} = \sum_{\ell=1} b_\ell \xi^\ell$$

where

$$n_0 = \frac{1}{\lambda^3} = \int \frac{d^3k}{(2\pi)^3}\, e^{-\beta E(k)}$$
$$\xi = e^{\mu/T}$$
$$b_1 = 1.$$

Given the density

$$n = \frac{\partial P}{\partial \mu} = \xi \frac{\partial}{\partial \xi} \beta P,$$

the virial expansion corresponds to an expansion of P in terms of n:

$$\frac{P}{nT} = \sum_{\ell=1} a_\ell \left(\frac{n}{n_0}\right)^{\ell-1}.$$

Show that

$$a_1 = 1 = b_1$$
$$a_2 = -b_2$$
$$a_3 = -2b_3 + 4b_2^2$$
$$a_4 = -3b_4 + 18 b_2 b_3 - 20 b_2^3.$$

b) The 2nd coefficient in an cluster expansion can be obtained by

$$b_2 = \frac{n_0}{2} \int d^3x_{12}\, f(r_{12})$$
$$f(r_{12}) = \left(e^{-\beta U(r_{12})} - 1\right).$$

Obtain an analytic expression for the hardcore potential:

$$U(r) = \begin{cases} \infty & r <= R \\ 0 & r > R. \end{cases}$$

Explain the sign of b_2 and its effect on pressure.

c) Derive the corresponding change in the virial expansion. Does a repulsive interaction increase or decrease the pressure? The jargon is that repulsive interaction gives a stiffer Equation of States.

d) Consider the Lennard-Jones potential

$$U(r) = \frac{a}{r^{12}} - \frac{b}{r^6}.$$

Plot the potential $U(r)$ and the Mayer's function $f(r)$ with some reasonable parameters. Show that at low temperatures b_2 is dominated by the long distance, attractive part of the potential. What happens when the temperature increases?

Q3 Method of auxiliary field.

Consider a 4-fermion interaction model

$$Z = \int D\psi D\bar{\psi} e^{\int \bar{\psi}(i\gamma\cdot\partial - m)\psi + G\,(\bar{\psi}\psi)^2}.$$

Note that the integral is over an Euclidean space time.

a) Prove the relation

$$e^{\int \frac{g^2}{2m_G^2}\,(\bar{\psi}\psi)^2} \propto \int D\sigma\, e^{\int \left(-g\sigma\bar{\psi}\psi - \frac{1}{2}m_G^2\sigma^2\right)}$$

and with this rewrite the 4-fermion interaction model as

$$Z \to \int D\sigma e^{\ln Z_F(M_F = m + g\sigma) - \int \frac{1}{2}m_G^2\sigma^2}.$$

where $Z_F(M_F)$ is the partition function for free fermions

$$\begin{aligned} \ln Z_F(M_F) &= \mathrm{tr}\ln\left(i\gamma\cdot\partial - M_F\right) \\ &= V_4 \int^{\Lambda} \frac{d^3k}{(2\pi)^3} 2\sqrt{k^2 + M_F^2} + \dots \end{aligned}$$

where $V_4 = \beta V$. Identify G in terms of g and m_G. Notice the similarity with Fermi coupling constant for weak interaction.

b) Suppose the functional integral over σ is dominated by a certain $\bar{\sigma}$, such that

$$Z = \int D\sigma \, e^{\ln Z_F(M_F = m + g\sigma) - \int \frac{1}{2} m_G^2 \sigma^2} \approx e^{-V_4 \Gamma(\bar{\sigma})}.$$

Derive a condition for $\Gamma(\bar{\sigma})$ based on the steepest descent. This is called the gap equation.

c) Solve the gap equation numerically at $T = 0$. (You may set $g = 1$.) Plot σ/Λ versus G for $m = 0$ and $m \neq 0$. Derive an analytic expression for the critical coupling for the former case. What is the order of the phase transition?

d) Find an explicit expression of the condensate via

$$\langle \bar{\psi}\psi \rangle = -\frac{\partial}{\partial m} \frac{\ln Z}{V_4}$$

Verify that the condensate is negative. In the model, one can compute it via

$$\begin{aligned} -n_S = \langle \bar{\psi}\psi \rangle &= \frac{\partial}{\partial m} \Gamma(\langle \sigma \rangle, m) \\ &= \frac{\partial}{\partial m} \Gamma(\langle \sigma \rangle, m) + \left(\frac{\partial}{\partial \sigma} \Gamma(\sigma, m) \frac{\partial \sigma}{\partial m} \right) \Big|_{\sigma = \langle \sigma \rangle}. \end{aligned}$$

Explain why the second term will not contribute and relate the condensate to σ. (This is model dependent!)

Q4 P-wave Resonance.

a) Revisit Q3 in HW6 1st sem. This was a very crude model of a resonance. A QFT-motivated approach would suggest

$$f(s) = K \frac{1}{s - m_{\text{bare}}^2 - \Sigma(s)}.$$

Note that Σ is the self energy and is complex. The proportionality constant K is irrelevant but can be worked out:

$$\begin{aligned} K &= \frac{2\text{Im}\Sigma(s)}{2q(s)} \\ q(s) &= \frac{1}{2} \sqrt{s} \sqrt{1 - \frac{(m_1 + m_2)^2}{s}} \sqrt{1 - \frac{(m_1 - m_2)^2}{s}}. \end{aligned}$$

The self energy of a P-wave resonance, after a (rather tedious) QFT calculation (with some additional manipulations), gives

$$\Sigma_R = -\frac{g^2}{8\pi^2} \int_0^1 dx\, \Delta \,\ln \Delta$$
$$\Delta = x m_1^2 + (1-x) m_2^2 - x(1-x)s - i\delta.$$

Show that the imaginary part reads

$$\mathrm{Im}\Sigma_R = -\frac{4}{3} \times \left(\frac{1}{2}\, g^2 \,\frac{q^3}{4\pi\sqrt{s}}\, \theta(\sqrt{s} - m_1 - m_2) \right).$$

(Compare with Q3 HW1 sem02, a $q(s)^2$ factor naturally emerges!)

b) Adjust the two parameters: m_{bare} and g^2 such that the model qualitatively describes the $\Delta(1232)$ resonance. Work out an explicit expression for the phase shift (in terms of Σ) and plot the result against $\sqrt{s}$. (use atan2)